

# Complete determination of the orbital parameters of a system with $N + 1$ bodies using a simple Fourier analysis of the data

Alexandre C.M. Correia

Departamento de Física da Universidade de Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal. [correia@ua.pt](mailto:correia@ua.pt)

**Summary.** Here we show how to determine the orbital parameters of a system composed of a star and  $N$  companions (that can be planets, brown-dwarfs or other stars), using a simple Fourier analysis of the radial velocity data of the star. This method supposes that all objects in the system follow keplerian orbits around the star and gives better results for a large number of observational points. The orbital parameters may present some errors, but they are an excellent starting point for the traditional minimization methods such as the Levenberg-Marquardt algorithms.

## 1 Radial Velocity

The radial velocity of a star with  $N$  companions is given by  $v(t) = \gamma + v_0(t)$ , where  $\gamma$  is a drift due to the global shift of the system center of mass, and [1]

$$v_0(t) = \sum_{j=1}^N K_j (e_j \cos \omega_j + \cos(\omega_j + \nu_j)) . \quad (1)$$

For each companion  $j$ ,  $e_j$  is the eccentricity,  $\omega_j$  the longitude of the perihelium,  $\nu_j = \nu_j(t)$  the true longitude of the date and

$$K_j = n_j a_j \frac{m_j}{\mathcal{M}} \sin I_j (1 - e_j^2)^{-1/2} \quad (2)$$

the amplitude of radial velocity variations.  $n_j$  is the mean motion,  $a_j$  the semi-major axis,  $I_j$  the inclination of the orbital plane with respect to the line of sight,  $m_j$  the mass and  $\mathcal{M}$  the total mass of the system. The orbital period of each companion is given by  $P_j = 2\pi/n_j$ .

### 1.1 Elliptic Expansions

There is no explicit expression for the true anomaly  $\nu_j(t)$ , but making use of the Kepler equation we can expand it in power series of  $e_j$  such that [2]:

$$e^{i\nu_j} = \sum_{k=-\infty}^{+\infty} C_k(e_j) e^{ikM_j} , \quad (3)$$

where  $M_j = n_j(t - T_{0j})$  is the mean anomaly,  $T_{0j}$  the date for the passage at the perihelium and

$$C_k(e_j) = \frac{1}{2\pi} \int_0^{2\pi} \left( \cos E - e_j + i\sqrt{1 - e_j^2} \sin E \right) e^{-ik(E - e_j \sin E)} dE . \quad (4)$$

To the fifth order in eccentricity,  $C_k(e_j)$  for  $k = 1$  and  $k = 2$  becomes:

$$C_1(e_j) = \left( 1 - \frac{9}{8}e_j^2 + \frac{25}{192}e_j^4 \right) + i \left( 1 - \frac{7}{8}e_j^2 + \frac{17}{192}e_j^4 \right) , \quad (5)$$

$$C_2(e_j) = \left( 1 - \frac{4}{3}e_j^2 + \frac{3}{8}e_j^4 \right) e_j + i \left( 1 - \frac{7}{6}e_j^2 + \frac{1}{3}e_j^4 \right) e_j . \quad (6)$$

Replacing expression (3) in (1) we can finally rewrite the radial velocity of the star as the real part of:

$$v_0(t) = \sum_{j=1}^N K_j e^{i\omega_j} \sum_{k=1}^{+\infty} C_k(e_j) e^{-ikn_j T_{0j}} e^{ikn_j t} . \quad (7)$$

## 2 Fourier Analysis

In this paper we will use an ordinary FFT transform of the radial velocity,

$$F(\phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} v(t) e^{-i\phi t} dt , \quad (8)$$

but other frequency analysis that make use of weight functions to ensure a better convergence with the data are possible. However, the calculus become more complicated, and harder to follow. Notice also that in the real case we cannot compute the FFT using the previous expression, because we are restricted to discrete number of observations  $N_{\text{obs}}$  in a time span  $[0, T]$ . Thus,

$$F(\phi) \approx \frac{1}{T} \int_0^T v(t) e^{-i\phi t} dt \approx \frac{1}{T} \sum_{k=2}^{N_{\text{obs}}} v_k e^{-i\phi t_k} (t_k - t_{k-1}) , \quad (9)$$

where  $v_k$  is the star radial velocity measured for the date  $t_k$ .

### 2.1 Determination of $\gamma$

Replacing  $v(t) = \gamma + v_0(t)$  in expression (8) with  $\phi = 0$  we get:

$$\gamma = F(0) . \quad (10)$$

It is then possible to have an estimation of  $\gamma$  using  $\phi = 0$  in expression (9). Once we have  $\gamma$  it is preferable to subtract its value from the data  $v_k$  and then continue the Fourier analysis (Eqs.8,9) with the expression of  $v_0(t)$  (Eq.7).

## 2.2 Determination of $n_j$

The orbital frequency  $n_1$  corresponding to the companion with largest amplitude  $K_1$  is given by the frequency  $\phi$  corresponding to the highest peak in the power spectrum, that is,

$$n_1 : \quad \forall \phi, \quad |F(n_1)| \geq |F(\phi)|. \quad (11)$$

After finding  $n_1$  it is easy to determine the remaining orbital parameters (see next section). Once the orbit of the first companion is completely established, it is recommended to subtract its contribution from the data  $v_k$  and then continue the Fourier analysis (Eqs.8,9) with the expression of

$$v_0 - K_1 (e_1 \cos \omega_1 + \cos(\omega_1 + \nu_1)) . \quad (12)$$

We then have to repeat this procedure for all the other  $N - 2$  remaining companions of the star. Thus, the  $n_j$  orbital frequencies are always given by the highest peak in the spectrum (Eq.11) after subtracting the signal from the already detected companions (Eq.12).

## 2.3 Determination of the remaining orbital parameters

Replacing expression (7) in (8) with  $\phi = n_j$  and  $\phi = 2n_j$  we have

$$F(n_j) = K_j e^{i\omega_j} C_1(e_j) e^{-in_j T_{0j}}, \quad (13)$$

$$F(2n_j) = K_j e^{i\omega_j} C_2(e_j) e^{-i2n_j T_{0j}}, \quad (14)$$

where the quantities  $F(n_j)$  and  $F(2n_j)$  can be computed from the data using expression (9). Multiplying the above expressions by their conjugates, we get

$$|F(n_j)| = K_j |C_1(e_j)| \quad \text{and} \quad |F(2n_j)| = K_j |C_2(e_j)|, \quad (15)$$

which gives an implicit condition for the eccentricity,

$$f(e_j) = \frac{|C_2(e_j)|}{|C_1(e_j)|} = \frac{|F(2n_j)|}{|F(n_j)|}, \quad (16)$$

where  $e_j$  can be determined using the bisection or the Newton's method[3]. After determining  $e_j$  it is now straightforward to compute  $K_j$  from Eqs.(15):

$$K_j = \frac{|F(n_j)|}{|C_1(e_j)|} = \frac{|F(2n_j)|}{|C_2(e_j)|}. \quad (17)$$

From expressions (13) and (14) we finally have

$$e^{in_j T_{0j}} = \frac{F(n_j) C_2(e_j)}{F(2n_j) C_1(e_j)} \quad (18)$$

and

$$e^{i\omega_j} = \frac{F(n_j)}{K_j C_1(e_j)} e^{in_j T_{0j}} = \frac{F^2(n_j) C_2(e_j)}{K_j F(2n_j) C_1^2(e_j)}. \quad (19)$$

### 3 Conclusion

For a single companion of a star, we are able to determine its orbital parameters directly from the observational data by computing the FFTs for three different frequencies, namely  $F(0)$ ,  $F(n)$  and  $F(2n)$ . We chose  $n$  and  $2n$ , but according to expression (7) we could have chosen any frequency multiple of  $n$ . However, unless the eccentricity is extremely high, this two frequencies correspond to the highest peaks produced by the companion in the spectrum and are therefore easier to identify. Moreover, if the eccentricity is close to zero (which is often the case for “hot Jupiters” and close binaries),  $F(kn) \approx 0$ , except for  $k = 0$  and  $k = 1$ . In this case  $\omega_j$  and  $T_{0j}$  cannot be determined, but it is still possible to establish the position of the planet in the orbit  $\lambda_j = \omega_j - n_j T_{0j}$  as

$$e^{i\lambda_j} = \frac{F(n_j)}{K_j C_1(e_j)} . \quad (20)$$

The orbital parameters determined with our method present errors that are proportional to the precision of the instrument and inversely proportional to the number of data points, since a large number of points increases the convergence between expressions (8) and (9). The agreement between the Fourier parameters and the true parameters can be increased if we perform a  $\chi^2$  minimization after determining the orbit of each companion. This procedure should be fast using a standard method such as a Levenberg-Marquardt algorithm[3], since the Fourier parameters are already close to the minimum value of  $\chi^2$ . Even though the FFT method is established for keplerian orbits, it also works on realistic systems for which planet-planet interactions are weak. Indeed, this method has already been tested with success in the determination of the orbital parameters of three different planetary systems[4, 5, 6], where we obtained the same results as other classical alternative methods.

### References

1. R.W. Hilditch: *An Introduction to Close Binary Stars*, (Cambridge University Press 2001)
2. C.D. Murray, S.F. Dermott: *Solar System Dynamics*, (Cambridge University Press 1999)
3. W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery: *Numerical recipes in FORTRAN*, (Cambridge University Press 1992, 2nd Ed.)
4. A.C.M. Correia, S. Udry, M. Mayor et al: *Astron. Astrophys.* **440**, 751 (2005)
5. C. Lovis, M. Mayor, F. Pepe et al: *Nature* **441**, 305 (2006)
6. F. Pepe, A.C.M. Correia, M. Mayor et al: *Astron. Astrophys.* **462**, 769 (2007)